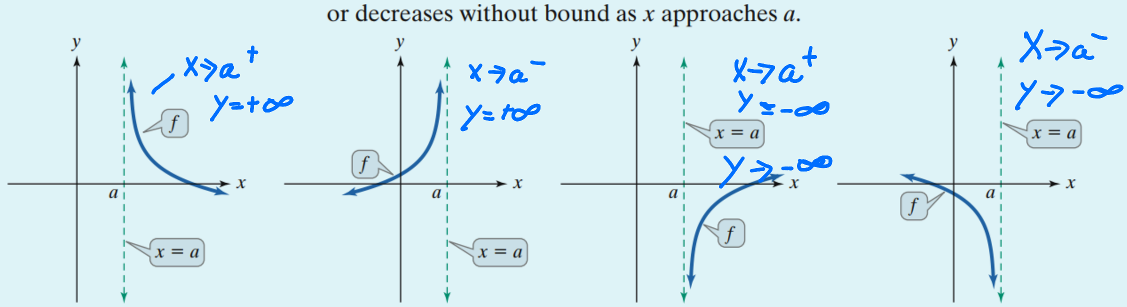
Period 4, Nov 19, 2024

Arrow Notation

Symbol	Meaning
$x \to a^+$	x approaches a from the right.
$x \to a^-$	x approaches a from the left.
$x \to \infty$	x approaches infinity; that is, x increases without bound.
$x \to -\infty$	x approaches negative infinity; that is, x decreases without bound.

Definition of a Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function f if $f(x)$ increases or decreases without bound as x approaches a.



As $x \to a^+$, $f(x) \to \infty$. As $x \to a^-$, $f(x) \to \infty$. As $x \to a^+$, $f(x) \to -\infty$. As $x \to a^-$, $f(x) \to -\infty$.

Thus, as x approaches a from either the left or the right, $f(x) \to \infty$ or $f(x) \to -\infty$.

Locating Vertical Asymptotes

If $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function in which $p(x)$ and $q(x)$ have no common factors and a is a zero of $q(x)$, the denominator, then $x = a$ is a vertical asymptote of the graph of f.

Determine whether the following statement "makes sense" or "does not make sense" and explain your reasoning.

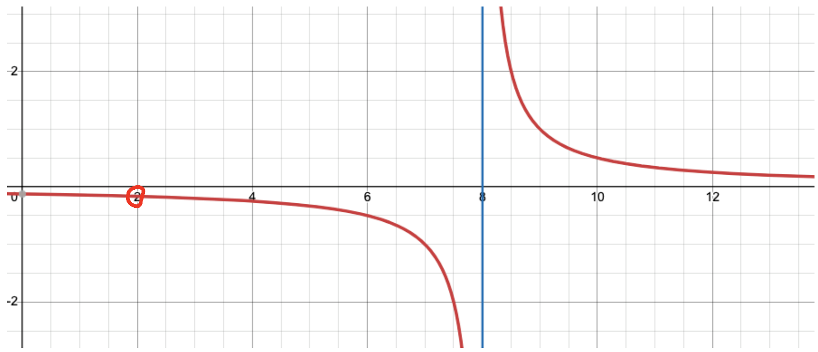
The function $y = \dfrac{x-2}{(x-2)(x-8)}$ has vertical asymptotes at $x = 2$ and $x = 8$.

$$\frac{2-2}{(2-2)(2-8)} = \frac{0}{0 \cdot -6} = \frac{0}{0} = \emptyset$$

hole

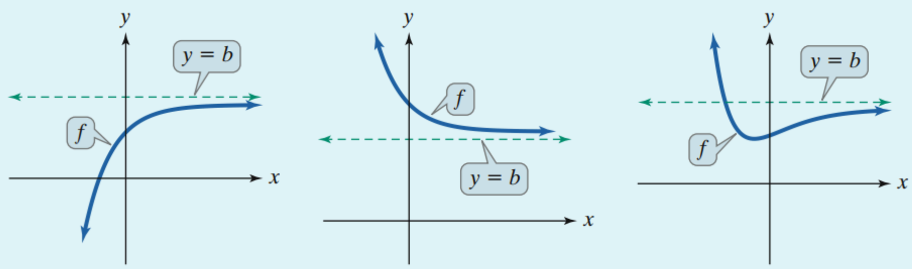
$$\frac{8-2}{(8-2)(8-8)} = \frac{6}{6 \cdot 0} = \frac{6}{0} = \emptyset$$

asymptote



Definition of a Horizontal Asymptote

The line $y = b$ is a **horizontal asymptote** of the graph of a function f if $f(x)$ approaches b as x increases or decreases without bound.



As $x \to \infty$, $f(x) \to b$. As $x \to \infty$, $f(x) \to b$. As $x \to \infty$, $f(x) \to b$.

Locating Horizontal Asymptotes

Let f be the rational function given by

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}, \quad a_n \neq 0, b_m \neq 0.$$

The degree of the numerator is n. The degree of the denominator is m.

1. If $n < m$, the x-axis, or $y = 0$, is the horizontal asymptote of the graph of f.
2. If $n = m$, the line $y = \frac{a_n}{b_m}$ is the horizontal asymptote of the graph of f.
3. If $n > m$, the graph of f has no horizontal asymptote. $x \to \infty$

$$F(x) = \frac{5x^7 - 12x^4 + 10{,}000{,}000x^3 + 17}{2x^8 + 9} \approx \frac{5x^7}{2x^8} = \frac{5}{2x} = 0$$

as $x \to \infty$
$y \to 0$
$x \to -\infty$
$y \to 0$

$F(x) = \dfrac{5x^7 - 12x^4 + 10{,}000{,}000x^3 + 17}{3x^7 - 10{,}000{,}000{,}000x^2} \approx \dfrac{5x^7}{3x^7} = \dfrac{5}{3}$

as $x \to \infty$
$y = \frac{5}{3}$
$x \to -\infty$
$y = \frac{5}{3}$

$F(x) = \dfrac{5x^7 - 12x^4 + 100x}{3x^6 + 12x} = \dfrac{5x^7}{3x^6} = \dfrac{5x}{3} = \infty$

as $x \to \infty$
$y \to \infty$
$x \to -\infty$
$y \to -\infty$

Strategy for Graphing a Rational Function

The following strategy can be used to graph

$$f(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions with no common factors.

1. Determine whether the graph of f has symmetry.

$$f(-x) = f(x): \quad y\text{-axis symmetry}$$
$$f(-x) = -f(x): \quad \text{origin symmetry}$$

2. Find the y-intercept (if there is one) by evaluating $f(0)$.
3. Find the x-intercepts (if there are any) by solving the equation $p(x) = 0$.
4. Find any vertical asymptote(s) by solving the equation $q(x) = 0$.
5. Find the horizontal asymptote (if there is one) using the rule for determining the horizontal asymptote of a rational function.
6. Plot at least one point between and beyond each x-intercept and vertical asymptote.
7. Use the information obtained previously to graph the function between and beyond the vertical asymptotes.

Graph: $f(x) = \dfrac{2x - 1}{x - 1}$.

NOT The Same
no y-axis sym

① $F(-x) = \dfrac{2(-x)-1}{-x-1} = \dfrac{-2x-1}{-x-1} = \dfrac{-1(2x+1)}{-1(x+1)}$

$-F(x) = -\dfrac{1}{1}\left(\dfrac{2x-1}{x-1}\right) = \dfrac{-2x+1}{x-1}$

NOT The Same
No origin sym

② $F(0) = \dfrac{2(0)-1}{0-1} = \dfrac{-1}{-1} = 1$

③ $(x-1)\,0 = \dfrac{2x-1}{\cancel{x-1}}\cancel{(x-1)} \Rightarrow 0 = 2x - \cancel{1} \Rightarrow 1 = 2x \Rightarrow x = \frac{1}{2}$
$+1 \quad\quad \cancel{+1}$

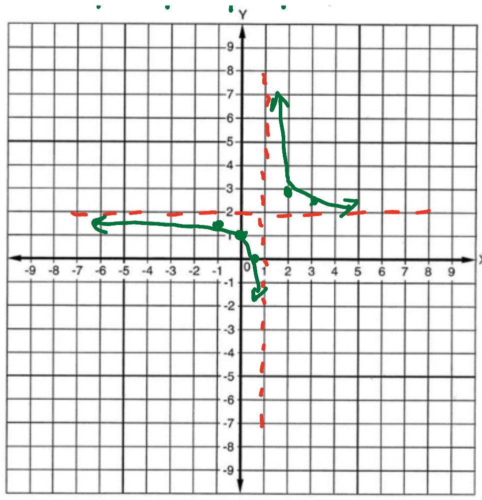
④ verTical asy

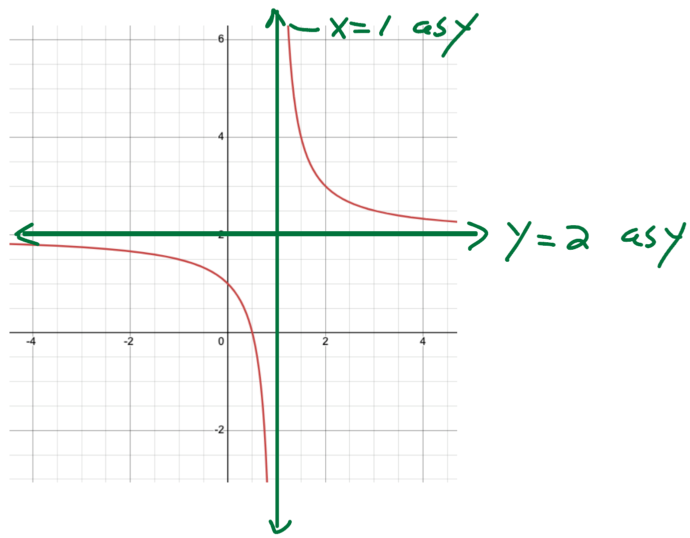
$\dfrac{2x-1}{x-1}$ $\quad$ $x-1=0$
$x=1$

⑤ $x \to \infty$
and
$x \to -\infty$

$F(x) = \dfrac{2x-1}{x-1} \approx \dfrac{2x}{x} = 2 = y$ = HorizonTail asy

x	0	$\frac{1}{2}$	1	-1	2	3
y	1	0	$\varnothing$	$\frac{3}{2}$	3	$\frac{5}{2}$





$$f(x) = \frac{3x^2}{x^2-4} = \frac{3x^2}{(x+2)(x-2)}$$

① $F(-x) = \frac{3(-x)^2}{(-x)^2-4} = \frac{3x^2}{x^2-4}$ $F(x) = F(-x)$ y-axis sym
Even

④ Vertical asy
$F(x) = \frac{3x^2}{(x+2)(x-2)}$

$x = -2,\ 2$

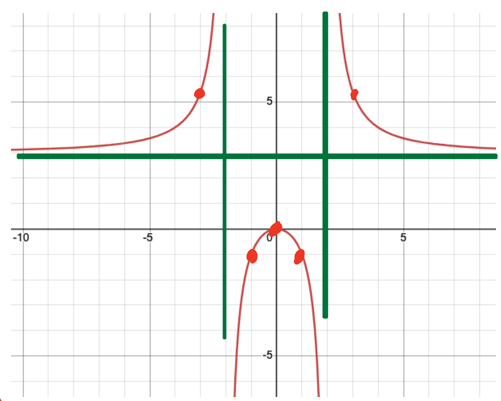
② $F(0) = \frac{3(0)^2}{0^2-4} = \frac{0}{-4} = 0$

③ $\cancel{(x^2-4)}\frac{3x^2}{\cancel{x^2-4}} = 0\,(x^2-4) \Rightarrow 3x^2 = 0$
$x = 0$

⑤ Horizontal asy $x \to \infty$
$F(x) \to \frac{3x^2}{x^2} = 3 = y$

6.

x	0	-2	2	1	-1	3	-3
y	0	∅	∅	-1	-1	$\frac{27}{5}$	$\frac{27}{5}$



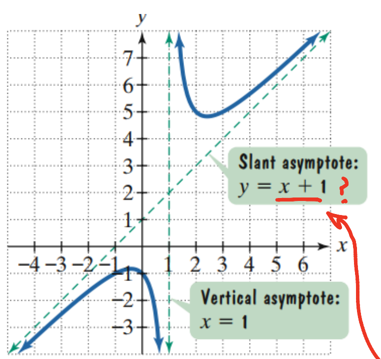


Figure 2.41 The graph of $f(x) = \frac{x^2+1}{x-1}$ with a slant asymptote

$$f(x) = \frac{x^2+1}{x-1},$$

$$\begin{array}{r} x+1 \\ x-1\,\overline{)\,x^2+0x+1} \\ -(x^2-x) \\ \hline 0+x+1 \\ -(x-1) \\ \hline 0 + \boxed{2}R \end{array}$$

$x+1+\frac{2}{x-1}$ $x \to \infty$ $x+1+\frac{2}{x-1} \approx (x+1)$

Applications

There are numerous examples of asymptotic behavior in functions that model real-world phenomena. Let's consider an example from the business world. The **cost function**, C, for a business is the sum of its fixed and variable costs:

$$C(x) = (\text{fixed cost}) + cx.$$

Cost per unit times the number of units produced, x

The **average cost** per unit for a company to produce x units is the sum of its fixed and variable costs divided by the number of units produced. The **average cost function** is a rational function that is denoted by $\overline{C}$. Thus,

Car Rental Problem

Cost of producing x units: fixed plus variable costs

$$\overline{C}(x) = \frac{(\text{fixed cost}) + cx}{x}.$$

Number of units produced

Use long division to rewrite the equation $g(x) = \frac{5x+6}{x+1}$ in the form quotient, plus remainder divided by divisor. Then use this form of the function's equation and transformations of $f(x) = \frac{1}{x}$ to graph g.

$$\begin{array}{r} 5 \\ x+1 \overline{)\, 5x+6} \\ -(5x+5) \\ \hline 0 \quad \boxed{1}\,R \end{array}$$

$$\frac{5x+6}{x+1} = 5 + \frac{1}{x+1}$$

$$F(x) = \frac{1}{x}$$

$$g(x) = \frac{1}{x+1} + 5$$

Left 1

Up 5

Divide using synthetic division.

$$\left(2x^2 - 8x - 4x^3 + x^4\right) \div (4 + x)$$

$x^4 - 4x^3 + 2x^2 - 8x + 0$

$$\begin{array}{r|rrrrr} -4 & 1 & -4 & 2 & -8 & 0 \\ & & -4 & 32 & 136 & -512 \\ \hline & 1 & -8 & 34 & 128 & \boxed{-512} \end{array}$$

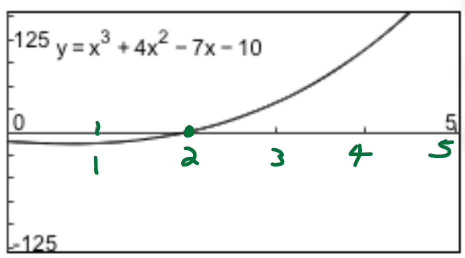
$x^3 - 8x^2 + 34x + 128 - \dfrac{512}{x+4}$

$\left(2x^2 - 8x - 4x^3 + x^4\right) \div (4 + x) = \square$

(Simplify your answer.)

Use the graph to determine a solution of the equation. Use synthetic division to verify that this number is a solution of the equation. Then solve the polynomial equation.

$$x^3 + 4x^2 - 7x - 10 = 0$$



[0, 5, 1] by [−125, 125, 25]

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -7 & -10 \\ & & 2 & 12 & 10 \\ \hline & 1 & 6 & 5 & \boxed{0} \end{array}$$

Which solution of the equation can be determined from the graph?

The solution is $x = 2$

$x^2 + 6x + 5$

$(x-2)(x+5)(x+1)$

$x = -5$

$x = -1$

Use the table to determine a solution of the equation. Use synthetic division to verify that this number is a solution of the equation. Then solve the polynomial equation.

$40x^3 - 53x^2 + 14x - 1 = 0$

X	$Y_1 = 40x^3 - 53x^2 + 14x - 1$
−3	−1600
−2	−561
−1	−108
0	−1
1	0
2	135
3	644

$F(1) = 0$

$$\begin{array}{c|cccc} 1 & 40 & -53 & 14 & -1 \\ & & 40 & -13 & 1 \\ \hline & 40 & -13 & 1 & 0 \end{array}$$

Which solution of the equation can be determined from the table?

The solution is 1.

$(x-1)(\underbrace{40x^2 - 13x + 1}) = 0$

40

$-8 \quad -5$

$\underbrace{40x^2 - 8x} \underbrace{-5x + 1}$

$8x(5x-1) - 1(5x-1)$

$(5x-1)(8x-1)$

What are the other solutions of the equation?

The set of the other solutions is { }. $\frac{1}{5}, \frac{1}{8}$

(Use a comma to separate answers as needed.)

$5x - 1 = 0$
$+1 \quad +1$
$\frac{5x}{5} = \frac{1}{5}$
$x = \frac{1}{5}$

$8x - 1 = 0$
$+1 \quad +1$
$\frac{8x}{8} = \frac{1}{8}$
$x = \frac{1}{8}$

Solve the equation $x^3 - 5x^2 + 2x + 8 = 0$ given that -1 is a zero of $f(x) = x^3 - 5x^2 + 2x + 8$.

$(x+1)(x-2)(x-4) = 0$

The solution set is {−1,4,2}. (Use a comma to separate answers as needed.)

$$\begin{array}{r} x^2 - 6x + 8 \\ x+1 \overline{) x^3 - 5x^2 + 2x + 8} \\ -(x^3 + x^2) \\ \hline 0 \quad -6x^2 + 2x \\ -(-6x^2 - 6x) \\ \hline 0 \quad +8x + 8 \\ -(8x + 8) \\ \hline 0 \end{array}$$

$(x+1)(\underbrace{x^2 - 6x + 8}) = x^3 - 5x^2 + 2x + 8 = 0$

$1 \cdot 8 = 8$

$-2 + -4 = -6$

$\underbrace{x^2 - 2x} \underbrace{-4x + 8}$

$x(x-2) - 4(x-2) = (x-2)(x-4)$

Solve the equation $28x^3 + 88x^2 - 37x - 7 = 0$ given that $-\frac{7}{2}$ is a zero of $f(x) = 28x^3 + 88x^2 - 37x - 7$.

The solution set is $\left\{ -\frac{7}{2}, -\frac{1}{7}, \frac{1}{2} \right\}$. (Use a comma to separate answers as needed.)

$$\begin{array}{r|rrrr} -\frac{7}{2} & 28 & 88 & -37 & -7 \\ & & -98 & 35 & +7 \\ \hline & 28 & -10 & -2 & \boxed{0} \end{array}$$

$28 \cdot -\frac{7}{2} = -98$

$-10 \cdot -\frac{7}{2} = 35$

$-\frac{7}{2} \cdot -2 = 7$

$(x+\frac{7}{2})(28x^2-10x-2)$

$(x+\frac{7}{2})(2)(14x^2-5x-1)$ $\Rightarrow$ $(x+\frac{7}{2})(2)(\underbrace{14x^2-7x}+\underbrace{2x-1})$

$14 \cdot -1 = -14$

$-7+2=-5$

$7x(2x-1)+1(2x-1)$

$(x+\frac{7}{2})(2)(2x-1)(7x+1)$

The function $f(x) = \frac{x+16}{x-4}$ is one-to-one. For the function,

$(y-4)x = \frac{y+16}{\cancel{y-4}} \cdot \cancel{(y-4)}$

a. Find an equation for $f^{-1}(x)$, the inverse function.

$\cancel{xy} - 4x = y + \cancel{16}$

$\cancel{-xy} \quad -16 \qquad -\cancel{16} \quad -xy$

b. Verify that your equation is correct by showing that $f\left(f^{-1}(x)\right) = x$ and $f^{-1}(f(x)) = x$.

$-4x - 16 = y - xy$

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a. Select the correct choice below and, if necessary, fill in the answer box to complete your choice.
(Simplify your answer. Use integers or fractions for any numbers in the expression.)

$\frac{-4x-16}{1-x} = \frac{y(\cancel{1-x})}{\cancel{1-x}}$

$\frac{-1(4x+16)}{-1(x-1)}$

- A. $f^{-1}(x) = \boxed{\frac{4x+16}{x-1}}$, for $x \neq \boxed{1}$
- B. $f^{-1}(x) = \square$, for $x \leq \square$
- C. $f^{-1}(x) = \square$, for all x
- D. $f^{-1}(x) = \square$, for $x \geq \square$

Verify that the equation is correct.

$f\left(f^{-1}(x)\right) = f\left(\boxed{\frac{4x+16}{x-1}}\right)$	and	$f^{-1}(f(x)) = f^{-1}\left(\boxed{\frac{x+16}{x-4}}\right)$	Substitute.
$= \boxed{x}$	and	$= \boxed{x}$	Simplify.

The equation is verified.

$F(x) = \frac{x+16}{x-4}$

$$F\left(\frac{4x+16}{x-1}\right) = \frac{\frac{4x+16}{x-1} + \frac{16(x-1)}{1(x-1)}}{\frac{4x+16}{x-1} - \frac{4(x-1)}{1(x-1)}} = \frac{\frac{4x+16}{x-1} + \frac{16x-16}{x-1}}{\frac{4x+16}{x-1} - \frac{4x-4}{x-1}} = \frac{\frac{4x+\cancel{16}+16x-\cancel{16}}{x-1}}{\frac{\cancel{4x}+16-\cancel{4x}+4}{x-1}}$$

$$x = \frac{\frac{20x}{x-1}}{\frac{20}{x-1}} = \frac{\cancel{20}x}{\cancel{(x-1)}} \cdot \frac{\cancel{(x-1)}}{\cancel{20}}$$

Divide using long division. State the quotient, q(x), and the remainder, r(x).

$$\frac{6x^4+27x^3-5x^2}{3x^2+2} = \frac{6x^4+27x^3-5x^2+0x+0}{3x^2+0x+2} = 2x^2+9x-3+\frac{-18x+6}{3x^2+2}$$

$$\begin{array}{r l}
 & 2x^2+9x-3 \\
3x^2+0x+2 \,\big) & 6x^4+27x^3-5x^2+0x+0 \\
 & -(6x^4+0x^3+4x^2) \\
 & 0 \quad 27x^3-9x^2+0x \\
 & -(27x^3+0x^2+18x) \\
 & 0-9x^2-18x+0 \\
 & -(-9x^2+0x-6) \\
 & 0x^2-18x+6 = R
\end{array}$$